Inner Product Space

Inner Product space :- Let X be a complex linear space. An inner product on X is a function $(\cdot,\cdot)$ i.e., $(\cdot,\cdot) : X \times X \to \mathbb{C}$ satisfying the following axioms :-

(i) $\overline{(x,y)} = (y,x)$

(ii) $(x,x) \geq 0$ & $(x,x) = 0 \Leftrightarrow x = 0$

(iii) $(ax+by, z) = a(x,z) + b(y,z)$

where a and b are complex numbers and $x, y \in X$.

An inner product space is a linear space with an inner product on it.

Result :- (i) $(x, y+z) = \overline{(y+z, x)}$

$$= \overline{(y,x)} + \overline{(z,x)}$$

$$= (x,y) + (x,z)$$

(ii) $(x, \alpha y) = \overline{(\alpha y, x)}$

$$= \overline{\alpha (y,x)}$$

$$= \bar{\alpha}\, \overline{(y,x)}$$

$$= \bar{\alpha}\, (x,y)$$

(iii) If $(x,z) = (y,z)$ $\forall$ z

then we have to prove $x = y$

Proof:- If $(x,z) = (y,z)$

$$\Rightarrow (x,z) - (y,z) = 0$$

$\Rightarrow (x-y, z) = 0 \quad \forall \ z \in X$

In particular for $z = x-y \in X$

$\Rightarrow (x-y, x-y) = 0$

$\Rightarrow x-y = 0$

$\Rightarrow x = y$

(iv) If $(x,y)=0 \ \forall \ y \in X$, Then $x=0$

Proof:- If $(x,y)=0 \ \forall \ y \in X$

In particular for $x=y$, $(x,x)=0 \Leftrightarrow x=0$

Shwartz's Inequality :-

Statement:- Prove that $|(x,y)| \leq (x,x)^{1/2}(y,y)^{1/2}$ for all $x,y \in X$, where X is I.P.S (Pre-Hilbert space)

Proof:- If $y=0$, then $|(x,y)| = |(x,0)| = 0$ & $(y,y)=0$

Then results holds.

Let us assume that $y \neq 0$

Now for any scalar λ, we have

$(x+\lambda y, x+\lambda y) \geq 0$

$\Rightarrow (x, x+\lambda y) + \lambda(y, x+\lambda y) \geq 0$

$\Rightarrow (x,x) + \bar{\lambda}(x,y) + \lambda(y,x) + \lambda\bar{\lambda}(y,y) \geq 0 \quad --- \ (*)$

Let us take $\lambda = -\dfrac{(x,y)}{(y,y)}$ $\left[\because y \neq 0 \Rightarrow (y,y) \neq 0 \right.$ Thus λ is defined$\left. \right]$

& $\bar{\lambda} = -\dfrac{(y,x)}{(y,y)}$

Put both these values in (*), we get

$$\Rightarrow (x,x) - \frac{(y,x)}{(y,y)}(x,y) - \frac{(x,y)}{(y,y)}(y,x) + \frac{(x,y)(y,x)}{(y,y)\cancel{(y,y)}}\cdot\cancel{(y,y)} \geq$$

$$\Rightarrow (x,x) - \frac{(y,x)(x,y)}{(y,y)} - \frac{(x,y)\cancel{(y,x)}}{\cancel{(y,y)}} + \frac{(x,y)\cancel{(y,x)}}{\cancel{(y,y)}} \geq 0$$

$$\Rightarrow (x,x) - \frac{(y,x)(x,y)}{(y,y)} \geq 0$$

$$\Rightarrow (x,x) - \frac{\overline{(x,y)}\,(x,y)}{(y,y)} \geq 0$$

$$\Rightarrow (x,x) - \frac{|(x,y)|^2}{(y,y)} \geq 0$$

$$\Rightarrow (x,x) \geq \frac{|(x,y)|^2}{(y,y)}$$

$$\Rightarrow (x,x)(y,y) \geq |(x,y)|^2$$

$$\Rightarrow |(x,y)| \leq (x,x)^{1/2}(y,y)^{1/2}$$

Hence the result

Also we can put $(x,x)^{1/2} = \|x\|$, $(y,y)^{1/2} = \|y\|$

Theorem:- Prove that every I.P.S is NLS.

Proof:- Let X be a I.P.S.

Define for $x \in X$

$$\|x\| = (x,x)^{1/2}$$

(i) Clearly $\|x\| \geq 0$

& $\|x\| = 0$

$$\Leftrightarrow (x,x)^{1/2} = 0$$

$$\Leftrightarrow (x,x) = 0$$

$$\Leftrightarrow x = 0$$

(ii) For any scalar α,

$$\|\alpha x\|^2 = (\alpha x, \alpha x)$$
$$= \alpha\bar{\alpha}(x, x)$$
$$= |\alpha|^2 (x, x)$$
$$= |\alpha|^2 \|x\|^2$$
$$\Rightarrow \|\alpha x\| = |\alpha| . \|x\|$$

(iii) Let $x, y \in X$, then

$$\|x+y\|^2 = (x+y, x+y)$$
$$= (x, x+y) + (y, x+y)$$
$$= (x,x) + (x,y) + (y,x) + (y,y)$$
$$= (x,x) + (x,y) + \overline{(x,y)} + (y,y)$$
$$= (x,x) + (y,y) + 2\,Re\,(x,y)$$
$$\leq (x,x) + (y,y) + 2\,|Re\,(x,y)|$$
$$\leq (x,x) + (y,y) + 2\,|(x,y)| \quad [\because |Re\,(z)|$$

Then by shwartz's inequality :

$$\|x+y\|^2 \leq (x,x) + (y,y) + 2\,(x,x)^{1/2}\,(y,y)^{1/2}$$
$$\leq \|x\|^2 + \|y\|^2 + 2\,\|x\|\,\|y\|$$
$$\leq (\|x\| + \|y\|)^2$$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$$

$\Rightarrow$ X is a NLS.

Theorem:- In an inner product space,

$|(x,y)| = \|x\| . \|y\|$ iff x & y are L.D.

Proof:- We have $\|x-\alpha y\|^2 = (x-\alpha y, x-\alpha y)$

$$= (x, x-\alpha y) - \alpha(y, x-\alpha y)$$

$$= (x,x) - \bar{\alpha}(x,y) - \alpha(y,x) + \alpha\bar{\alpha}(y,y)$$

Take $\alpha = \dfrac{(x,y)}{(y,y)}$ & $\bar{\alpha} = \dfrac{(y,x)}{(y,y)}$

Then, $(x,x) - \dfrac{(y,x)(x,y)}{(y,y)} - \dfrac{(x,y)}{(y,y)}(y,x) + \dfrac{(x,y)(y,x)}{(y,y)\cancel{(y,y)}}\cancel{(y,y)}$

$$\Rightarrow (x,x) - \frac{(y,x)(x,y)}{(y,y)} - \frac{(x,y)\cancel{(y,x)}}{\cancel{(y},y)} + \frac{(x,y)\cancel{(y},x)}{\cancel{(y},y)}$$

$$\Rightarrow (x,x) - \frac{\overline{(x,y)}\,(x,y)}{(y,y)}$$

$$= (x,x) - \frac{|(x,y)|^2}{(y,y)} \quad \text{---} \text{(i)}$$

Now suppose first x & y are L.D, then

$x = \alpha y$ i.e., L.H.S of (i) is zero

and so

$$(x,x) - \frac{|(x,y)|^2}{(y,y)} = 0$$

$$\Rightarrow (x,x) = \frac{|(x,y)|^2}{(y,y)}$$

$$\Rightarrow |(x,y)|^2 = (x,x)(y,y)$$

$$\Rightarrow |(x,y)| = (x,x)^{1/2}(y,y)^{1/2} = \|x\| . \|y\|$$

Conversely:-

If $|(x,y)| = \|x\|.\|y\|$

Then R.H.S on (i) is zero and therefore, L.H.S of (i) is also zero

i.e., $\|x-\alpha y\| = 0$ or $x = \alpha y$

$\Rightarrow$ x & y are Linearly dependent.

Parallelogram Law in a Pre-Hilbert space (IPS)

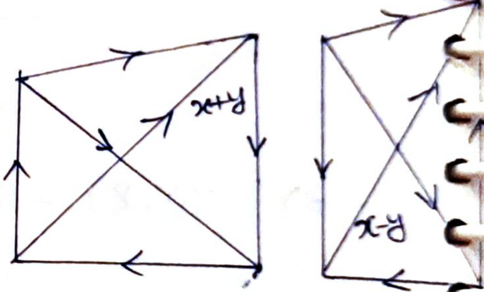
holds:- $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x,y \in$

Proof:- We have $\|x+y\|^2 + \|x-y\|^2 = (x+y, x+y) + (x-y, x-y)$

$= (x, x+y) + (y, x+y) + (x, x-y) - (y, x-y)$

$= (x,x) + \cancel{(x,y)} + \cancel{(y,x)} + (y,y) + (x,x) - \cancel{(x,y)}$

$- \cancel{(y,x)} + (y,y)$



$= 2(x,x) + 2(y,y)$

$= 2\|x\|^2 + 2\|y\|^2$

Theorem (Polarization Identity)

Statement:- In a Pre-Hilbert Space

$$(x,y) = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right]$$

Proof:- We have

$\|x+y\|^2 = (x+y, x+y)$

$= (x, x+y) + (y, x+y)$

$= (x,x) + (x,y) + (y,x) + (y,y)$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + (x,y) + (y,x)$

Now Replace y by $-y$, iy & $-iy$, we have

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - (x,y) - (y,x)$$

$$\& \ \|x+iy\|^2 = \|x\|^2 + \|y\|^2 - i(x,y) + i(y,x)$$

$$\& \ \|x-iy\|^2 = \|x\|^2 + \|y\|^2 + i(x,y) - i(y,x)$$

It follows that

$$(x,y) = \frac{1}{4}\Big[\cancel{\|x\|^2} + \cancel{\|y\|^2} + (x,y) + \cancel{(y,x)} - \cancel{\|x\|^2} - \cancel{\|y\|^2} + (x,y) + \cancel{(y,x)} + i\cancel{\|x\|^2} + i\cancel{\|y\|^2} + (x,y) - \cancel{(y,x)} - i\cancel{\|x\|^2} - i\cancel{\|y\|^2} + (x,y) - \cancel{(y,x)} \Big]$$

$$(x,y) = \frac{1}{\cancel{4}}\left[\cancel{4}(x,y)\right]$$

$$(x,y) = (x,y)$$

Hence the result.

V. Imp.

Theorem Prove that Inner product is jointly continuous.

Proof:- Let $\{x_n\}$ & $\{y_n\}$ be any sequences in X s.that $x_n \to x$ and $y_n \to y$

To show: $(x_n, y_n) \to (x,y)$

Consider $|(x_n,y_n) - (x,y)| = |(x_n,y_n) - (x_n,y) + (x_n,y) - (x,y)|$

$$= |(x_n, y_n - y) + (x_n - x, y)|$$

$$\leq |(x_n, y_n - y)| + |(x_n - x, y)|$$

By shwartz's inequality

$$|(x_n,y_n) - (x,y)| \leq \|x_n\| . \|y_n - y\| + \|x_n - x\| . \|y\| \quad \text{--- (1)}$$

Now, $\big|\|x_n\| - \|x\|\big| \leq \|x_n - x\| \to 0$ as $n \to \infty$

$\Rightarrow \|x_n\| \to \|x\|$ as $n \to \infty$

$\Rightarrow \{\|x_n\|\}$ is convergent sequence.

and hence bounded

$\Rightarrow \|x_n\| < M$ for some +ve real number

Now taking limit as $n \to \infty$

Then from (i), we get

$|(x_n, y_n) - (x, y)| \to 0$ as $n \to \infty$

$\Rightarrow (x_n, y_n) \to (x, y)$

Def^n:- A complete Pre-Hilbert space is called a Hilbert space.

Thus a Banach space whose norm is generated by an inner product is called a Hilbert space

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Imp.

Result:- In an I.P.S every cauchy sequence is Bounded.

Proof:- Let $\{x_n\}$ be a cauchy sequence in I.P.S X.

Then by def^n of cauchy sequence for $\varepsilon(=1)$, $\exists$ a +ve integer N s. that

$\|x_n - x_m\| < 1 \quad \forall \quad n, m \geq N$

In particular for $m = N$

$\|x_n - x_N\| < 1 \quad \forall \quad n \geq N$

Now for $n \geq N$. Consider $\|x_n\| = \|x_n - x_N + x_N\|$

$\leq \|x_n - x_N\| + \|x_N\|$

$< 1 + \|x_N\|$

Let M be max. of $\|x_1\|, \|x_2\|, ---, \|x_{N-1}\|, 1 + \|x_N\|$

$\Rightarrow \|x_n\| \leq M \quad \forall \; n$

$\Rightarrow \{x_n\}$ is Bounded.